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LETTER TO THE EDITOR

Corner transfer matrix of a critical free Fermion system

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Abstract. In order to diagonalize the generator of the corner transfer matrix of a critical free-fermion 8-vertex system with magnetic field we derive an integral representation for the eigenvector components of the generator. At large N we numerically verify the predictions of conformal invariance for critical systems.

The predictions of conformal invariance [1,2] for critical two-dimensional systems have been verified so far for many geometries and configurations pertaining to both row-to-row [3] as well as corner transfer matrices [4]. However it has been noticed that in the presence of external fields some adjustment is necessary [5]. Here we study a critical free-fermion system of finite size [6] from the standpoint of its corner transfer matrix where we encounter a situation which differs from a case studied previously [7] and thus leads to some mathematical aspects which have not yet been treated.

To set up the framework let us recall that instead of dealing directly with the corner transfer matrix itself we seek to diagonalize its generator obtained from the original corner transfer matrix by the so-called Hamiltonian limit (see e.g. [7]). The quantum operator obtained is

$$L_0 = \sum_{n=1}^{N-1} \{ n(\sigma_n^x \sigma_{n+1}^x + \lambda \sigma_n^y \sigma_{n+1}^y) + h(2n-1)\sigma_n^z \} + h(N-1)\sigma_N^z \tag{1}$$

where λ is the so-called anisotropy parameter and h the magnetic field. L_0 can be interpreted as the Hamiltonian of a quantum XY spin chain with linearly increasing strength of the exchange coupling along the chain in a transverse magnetic field which is also increasing linearly along the chain.

Since this statistical system exhibits a critical behaviour for $\lambda = 1$ (the case treated in [7]) it is interesting to study the other critical line [6]

$$\lambda = 2h - 1. \tag{2}$$

We follow up the standard procedure of Lieb *et al* [8] which consists in fermionizing L_0 and diagonalizing the quadratic form in fermion operators. Basically this amounts

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to diagonalizing two matrices (which do not commute) and the components of the eigenvectors $\{\psi\} = (\psi_1, \ldots, \psi_N)$ and $\{\phi\} = (\phi_1, \ldots, \phi_N)$ have in the end to fulfil the recursion relations

$$(n-1)\psi_{n-1} + n\lambda\psi_{n+1} - h(2n-1)\psi_n = \varepsilon\phi_n \tag{3}$$

$$\lambda(n-1)\phi_{n-1} + n\phi_{n+1} - h(2n-1)\phi_n = \varepsilon\psi_n \tag{4}$$

with the end components obeying

$$(N-1)\psi_{N-1} - h(N-1)\psi_N = 0 \tag{5}$$

$$\lambda(N-1)\phi_{N-1} - h(N-1)\phi_N = 0. \tag{6}$$

In the special cases studied previously the resolution of this system has been facilitated enormously because one was able to identify (3) and (4) with the recursion relations of known polynomials such as Meixner, Gottlieb, Laguerre and Carlitz polynomials. The reason for this was that the recursion relations decoupled, resulting in tridiagonal recursions. Unfortunately in our case of (2) such an identification is no longer possible.

The method employed here makes use of the generating functions

$$\psi(t) = \sum_{n=1}^{\infty} t^{n-1}\psi_n \tag{7}$$

$$\phi(t) = \sum_{n=1}^{\infty} t^{n-1}\phi_n \tag{8}$$

which upon substitution in (3) and (4) yield the system of first-order differential equations

$$(t^2 + \lambda - 2ht)\psi' + (t-h)\psi = \varepsilon\phi \tag{9}$$

$$(\lambda t^2 + 1 - 2ht)\phi' + (\lambda t - h)\phi = \varepsilon\psi. \tag{10}$$

The solution† appears under the form of Meixner's generating functions [9], i.e.

$$\psi(t) = f(t)\exp(\varepsilon u(t,\lambda)) \tag{11}$$

$$\phi(t) = g(t)\exp(\varepsilon u(t,\lambda)) \tag{12}$$

with

$$f(t) = (t^2 + \lambda - 2ht)^{-1/2} \qquad g(t) = (\lambda t^2 + 1 - 2ht)^{-1/2} \tag{13}$$

and taking the condition of criticality $\lambda = 2h - 1$ already into account

$$u(t,\lambda) = \int_0^t \frac{\mathrm{d}x}{(1-x)\sqrt{(\lambda - x)(1-\lambda x)}}. \tag{14}$$

† The authors thank Professor Brian Davies for providing hints leading to this solution.

Here $\psi(t)$ and $\phi(t)$ do not generate the usual Meixner polynomials [10] because of the complexity of (14). In fact for $0 < \lambda < 1$ $u(t,\lambda)$ has the following expression

$$u(t,\lambda) = \frac{2}{(1-\lambda)}\left\{\cos^{-1}\sqrt{\frac{(\lambda-t)}{(1+\lambda)(1-t)}} - \cos^{-1}\sqrt{\frac{\lambda}{1+\lambda}}\right\} \quad (15)$$

for $t \leqslant \lambda$ and

$$u(t,\lambda) = \frac{2}{1-\lambda}\left\{\left(\frac{\pi}{2} - \cos^{-1}\sqrt{\frac{\lambda}{1+\lambda}}\right) - \mathrm{i}\left(\cosh^{-1}\sqrt{\frac{(1-\lambda t)}{(1+\lambda)(1-t)}} - \cosh^{-1}\sqrt{\frac{1}{1+\lambda}}\right)\right\} \quad (16)$$

for $t > \lambda$. As $\lambda \to 1$ $u(t,\lambda) \to u(t,1) = t/(1-t)$ consistent with [7], where we recover the Laguerre polynomials. As for $\lambda \to 0$ one should take the second form of $u(t,\lambda)$. Then it agrees with the limiting case ($\lambda = 0$, $h = 1/2$) of a special triangular Ising model [11].

Each set of polynomials $\{\psi_n(\varepsilon)\}$ and $\{\phi_n(\varepsilon)\}$ obeys a pentadiagonal recursion relation, if one eliminates appropriately in (3)–(6). Thus they are not necessarily orthogonal as Favard's theorem [10] states. In fact the special Meixner structure of the generating function does not automatically imply orthogonality as exemplified by a case in [12]. The issue of orthogonality must therefore be settled separately. In any case one may always use Cauchy's theorem to obtain integral representations for the polynomials. For example $\psi_n(\varepsilon)$ takes the form

$$\psi_n(\varepsilon) = \frac{1}{2\pi\mathrm{i}}\oint \frac{\mathrm{d}t}{t^{n+1}}\frac{\exp(\varepsilon u(t,\lambda))}{\sqrt{(t^2+\lambda-(\lambda+1)t)}}. \quad (17)$$

A more practical representation may be obtained with the following parametrization

$$\exp(\mathrm{i}\alpha/2) = \frac{1+\mathrm{i}\sqrt{\lambda}}{\sqrt{1+\lambda}} \qquad z = \frac{1-\lambda}{2}u \qquad x = \frac{2\varepsilon}{1-\lambda}$$

and choosing as contour in the z-plane a vertical strip limited by $z = \alpha/2$ on the right and $z = (\alpha-\pi)/2$ on the left, so that the mapping from t to z, given by (15), surrounds only the pole at the origin, i.e.

$$t = \tan z \tan(\alpha - z). \quad (18)$$

With this parametrization $\psi_n(x)$ may be written in terms of Fourier integrals

$$\psi_n(x) = \frac{2}{\pi\sqrt{1+\lambda}}\left\{\mathrm{e}^{\alpha x/2}\int_{-\infty}^{\infty}\mathrm{d}s\,\mathrm{e}^{\mathrm{i}xs}\frac{[\cos\alpha+\cosh 2s]^{n-1}}{[\cosh 2s-\cos\alpha]^n}\cosh s - \mathrm{i}\mathrm{e}^{(\alpha-\pi)x/2}\int_{-\infty}^{\infty}\mathrm{d}s\,\mathrm{e}^{\mathrm{i}xs}\frac{[\cosh 2s-\cos\alpha]^{n-1}}{[\cosh 2s+\cos\alpha]^n}\sinh s\right\}. \quad (19)$$

As $n \to \infty$, the second integral in (20) is vanishingly small and the asymptotic behaviour of $\psi_n(x)$ is essentially given by the first Fourier integral. The zeros of the

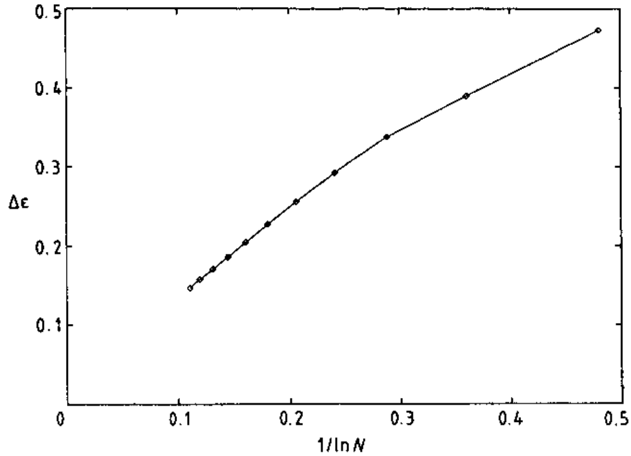

Figure 1. The spacing $\Delta\varepsilon$ between the two lowest lying eigenvalues of L_0 for $\lambda = \frac{1}{2}$ and $h = \frac{3}{4}$ as a function of $1/\ln N$ (lines are guides to the eye only).

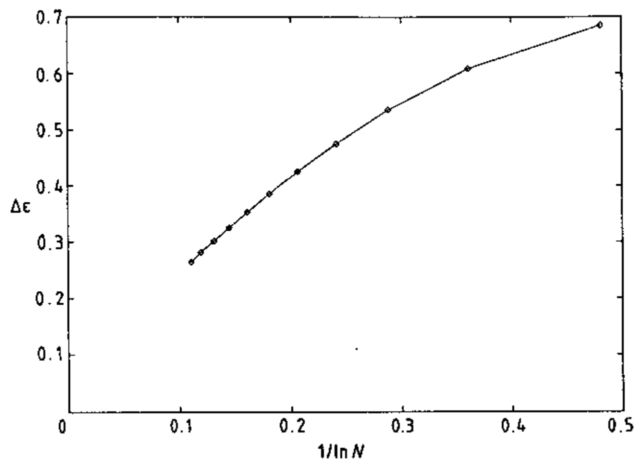

Figure 2. The spacing $\Delta\varepsilon$ between the two lowest lying eigenvalues of L_0 for $\lambda = 0$ and $h = \frac{1}{2}$ as a function of $1/\ln N$ (lines are guides to the eye only).

asymptotic expression of $\psi_n(x)$ determine the eigenvalues of the quantum operator L_0 for a large but finite system. We shall study this issue in detail later [13] and restrict ourselves for the moment to numerical results which confirm the critical behaviour expected for the corner transfer matrix of critical systems [2].

Note that a similar representation for $\phi_n(x)$ can be obtained if the factors $\cosh s$ and $-\mathrm{i}\sinh s$ are exchanged in (20).

We diagonalized the system (3)–(6) numerically for several values of the anisotropy parameter λ and the magnetic field h for system sizes $N = 2^m$ with $m = 3, \ldots, 13$. Conformal invariance predicts [ ] for the low-lying eigenvalues of L_0 that the levels are equidistantly spaced with the level spacing $\Delta\varepsilon$ vanishing logarithmically as the system becomes large

$$\Delta\varepsilon = \frac{2\pi}{\ln N}. \tag{20}$$

This behaviour, as is well known, may be modified by an amplitude which is proportional to the Fermi velocity and therefore depends on applied external fields, in our case the magnetic field h. This effect has already been discussed for the corner transfer matrix [7]. In order to observe the behaviour (21) numerically one has to treat sufficiently large chains, as we did, or to modify the chains by cutting off some sites at the left end of the chains [14]. We have plotted the spacing between the two lowest lying eigenvalues for two sets of values (h, λ) on the critical line (2), see the figures. Of course only the analytical treatment via the asymptotic evaluation of (20) can prove the correct h-dependence of the prefactor which modifies (21). From the numerical curves, however, we have evidence that the dependence is the same as in the case of $\lambda = 1$ and $h \leqslant 1$ [7].

To conclude we have confirmed the predictions of conformal invariance for the corner transfer matrix on the critical line $\lambda = 2h - 1$ numerically and demonstrated that an analytical treatment through the use of polynomials and their asymptotic evaluation should be feasible, but is more involved because of the pentadiagonal structure of the recursion relations in this case.

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